Interacting Many-Investor Models, Opinion Formation and Price Formation with Non-extensive Statistics.

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Abstract

We seek to utilize the nonextensive statistics to the microscopic modeling of the interacting many-investor dynamics that drive the price changes in a market. The statistics of price changes are known to be fit well by the Students-T and power-law distributions of the nonextensive statistics. We therefore derive models of interacting investors that are based on the nonextensive statistics and which describe the excess demand and formation of price.

1   Introduction

As is known from economics, the price of a security (for example) can be related to the law of supply and demand. That is to say, the excess demand is proportional to the price such that we can write approximately

$$x = \frac{1}{\lambda}\Delta\phi$$

(1)

and here λ is the market depth [4, 5]. In the past decade or so, there have been many models [4, 5] proposed that attempt to capture the dynamics and statistics of market participants. These range from minority game models [11], multi-agent models, and lattice super-spin models that encode the many degrees of freedom available to an interacting investor as the degrees of freedom of the variables and spins of the models. These models attempt to quantify the excess demand brought about by the mismatch of supply and demand between interacting investors in a market. The hallmark for the success of a model has been the ability of the model in reproducing the stylized facts of real markets. These are the heavy tails (power-law) of the distributions, anomalous (super) diffusion, and therefore statistical dependence (long-range correlations) of subsequent price changes.

Recently we reported on an application of the C.Tsallis nonextensive statistics to the S&P500 stock index [1, 3]. There we argued that the statistics are applicable to a broad range of markets and exchanges where anamolous (super) diffusion and 'heavy' tails of the distribution are present, as they are in the S&P500 [3]. In effect we have characterized the statistics of the price changes (the left hand side of Eq.(1)) as being well-modeled by the non-extensive statistics. We now seek to examine the demand-side of the equation in light of our recent findings that the non-extensive statistics models well the statistics of the price changes in real markets. As such, we will seek to outline a method by which one can obtain many-investor models within the context of the Tsallis nonextensive statistics. We will derive our specific models utilizing the well-known techniques of the maximum entropy approach [6, 8]. Let us briefly review the maximum entropy approach to be utilized here. The nonextensive, least-biased probability distribution function (PDF) $P(z,t)$ of an underlying observable $z(t)$ is obtained by maximizing an incomplete information theoretic measure equivalent to the Tsallis entropy S_q [1, 6, 8]

$$\langle S_q \rangle = S_q = -\frac{1}{1-q} \left(1 - \int P(z,t)^q \, dz \right).$$

$$(2)$$

Here $P(z,t)$ is the probability distribution function and will be shown to be of a power-law form, and is the degree of non-extensivity or equivalently the incompleteness of the information measure. The inverse of the normalization is the partition function $Z(t)$, and $\beta(t)$ is a Lagrange multiplier associated with the constraint(s). In order to build our model(s), we must specify the constraints to be utilized in our maximization procedure. These constraints will be the known (or assumed) observables of interest, and that are presumed to capture the deterministic behavior of our many-investor system. An interesting model for the investors is the model of investor bias and demand developed by Cont and Bouchaud and generalized to super-spins by Chowdhury [4, 5, 9]. This model assumes that the generalized spins are representative of the magnitude and direction of demand (the bias) of the investors. We will adopt this model, with suitable modifications, as a first approximation for characterizing the deterministic investor dynamics as observables from within the context of the nonextensive statistics. Following the work of Chowdhury et. al. let us define the demand function of a system of interacting investors as a classical Hamiltonian-like function in physics. This Hamiltonian will then be dependent on the magnitude of the demand, and the other degree of freedom, the direction of the demand or the bias. The bias in this model will be taken in this initial model to be discrete and will be represented as a spin. Also, for simplicity, let us initially assume that the magnitude is fixed. The N-investor interaction potential in the Hamiltonian then is taken to be made of discrete terms with simple constant ferromagnetic coupling strengths ($J_{ij} = J > 0$) and an anti-ferromagnetic coupling ($L_{ij} = L > 0$) which we treat in the global mean field sense of

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the Bornholdt model [9, 10] such that

$$(M = Nm, \ m = \frac{\langle \sigma_j \rangle}{N})$$

$$
\begin{aligned}
V &= -\sum_{i,j}^{N} J_{ij}\sigma_i\sigma_j + \sum_{i}^{N} L_i\sigma_i M \\
&= -J\sum_{i,j}^{N} \sigma_i\sigma_j + L\sum_{i}^{N} \sigma_i M
\end{aligned}
$$

(3)

The first potential term can be seen to be the usual form of a ferromagnetic spin-spin interaction with a coupling strength and models the herd-like, or collective opinion formation of investors. The spins represent discrete bias, though the generalization to a continuous degree of bias is straightforward, and can take on the values

$$
\sigma_i = \left\{
\begin{array}{l}
+1, \ \text{Bull} \ (\uparrow) \\
0, \ \text{Neutral} \ (0) \\
-1, \ \text{Bear} \ (\downarrow)
\end{array}
\right\},
$$

(4)

and therefore when two investors minimize their risk and both agree in their bias $\sigma_i\sigma_j = 1$ the overall energy is minimized. Note that in this simple model if one of two investors is neutral, the interaction term is zero. The second potential term is anti-ferromagnetic ($L > 0$) and models contrarian investor behavior as in the Bornholdt model. We can simplify the interactions by assuming a mean field approximation. Now let us assume that the bias fluctuates. This is a reasonable assumption, and leads us to the necessity of describing the averages of the fluctuations in terms of the mean and the variance. These moments can be written for the ith investor as q-parametrized averages [6, 8]

$$
\begin{aligned}
\langle \sigma - \bar{\sigma} \rangle_q &= \sum_{\sigma} (\sigma - \bar{\sigma}) \ P^q(\sigma) = 0, \\
\left\langle (\sigma - \bar{\sigma})^2 \right\rangle_q &= \sum_{\sigma} (\sigma - \bar{\sigma})^2 \ P^q(\sigma) = \rho_q^2 \ .
\end{aligned}
$$

(5)

In order to obtain a tractable solution that retains the essential interaction dynamics to lowest order, let us make a mean field approximation to our Hamiltonian. This will allow us to linearize the spin-spin interaction and will allow us to derive a closed form solution which, it is supposed, will allow us to qualify and approximately quantify the inherent behavior of the system. The full solution to this model would perforce involve the complicated spin-spin terms and will result in corrections to our solved form mean field solution. As stated then, the mean field solution will give us approximately the first order response of the system, given the

interactions. We then have (H_{mf} is the mean field Hamiltonian)

$$
\begin{aligned}
H_{mf} &= -J \sum_{i,j}^{N} \sigma_i \langle \sigma_j \rangle + L \sum_{i}^{N} \sigma_i M \\
&= -(J-L)M \sum_{i}^{N} \sigma_i, \\
m &= \frac{\langle \sigma_j \rangle}{N}. \text{ and } M = Nm
\end{aligned}
\tag{6}
$$

Here m is the average bias per investor and can be seen to be the analog to the average magnetization per particle. Next let us build into this model the open nature of a market. That is, the number of investors is not a constant in a market, and we will account for this fact by the inclusion of the number of investors as a further constraint (observable) in our model. As such the observable (per investor) to be included in our maximization procedure will be

$$
(h^i_{mf} = H^{i}_{mf}/N = -(J-L)m \ \sigma_i)
\tag{7}
$$

$$
K = h^i_{mf} - \mu,
\tag{8}
$$

and μ is a total investor number N constraint multiplier which from the usual thermodynamic analogies goes as the 'chemical potential'. We therefore can write for the ith investor the following observables to be included in our entropy maximization (and dropping the i sub- and superscripts)

$$
\begin{aligned}
\langle S \rangle_q &= S_q = -\frac{1}{1-q} \left(1 - \sum_{\sigma} P(\sigma)^q \right), \\
\langle K \rangle_q &= \sum_{\sigma} [-(J-L)m \ \sigma \ - \mu] \ P(\sigma)^q .
\end{aligned}
\tag{9}
$$

The maximum entropy approach then allows us to vary the entropy given the constraints such that

$$
\delta \langle S \rangle_q - \delta \left\{ \beta \langle K \rangle_q \right\} \equiv 0
\tag{10}
$$

and we obtain our least biased probability density function as a Tsallis non-extensive statistics power-law form

$$
P(\sigma) = \frac{\{1 + \beta(q-1)[-(J-L)m \ \sigma \ - \mu]\}^{\frac{-1}{q-1}}}{Z_q},
\tag{11}
$$

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and here the partition function is related to the normalization and is given by $Z_q = \sum \sigma P(\sigma)$

We now wish to examine the average bias in this model. Following the usual magnetic systems argument, the average bias can be written as (recall $\uparrow = +1, \downarrow = -1$)

$$m = n_\uparrow - n_\downarrow = \frac{N_\uparrow - N_\downarrow}{N},$$
(12)

The question then is how to obtain $(N \uparrow, N \downarrow)$ given that $N = N_+ + N_0 + N_-$. We can write as before

$$
\begin{aligned}
m &= \langle \sigma \rangle \\
&= \sum_\sigma \sigma \, P(\sigma) \\
&= P(\sigma = +1) \; - \; P(\sigma = -1)
\end{aligned}
$$
(13)

This expression can then be related to the average price change by the market depth and we obtain our desired result. That is, the Tsallis power-law statistical distribution for the price and price changes, as reported elsewhere [1] for stock market indices such as the S&P500 high frequency price data, is obtained from the individual investor bias distributions with a proportionality factor of market depth converting the excess demand to price in currency. We make use of the market depth λ and write (x is the price)

$$x = \frac{\langle \sigma \rangle}{\lambda}.$$
(14)

2   continuous spin model

We now wish to generalize beyond the limitations of the approximations we have built into the pure spin model. To do this, let us assume that the state vector for the system is of two dimensions, the magnitude and the bias. We will then work with continuum spins as in the Kosterlitz-Thouless Hamiltonian. Let us write down the observables for the interacting investors in the two dimensions (y= magnitude,θ=bias 'angle') of magnitude and bias as V. We then have

$$V = V_{int} + U_{ext} = -J \sum_{i,j}^{N} cos(\theta_i - \theta_j) + LM \sum_{i,j}^{N} cos(\theta_i)$$
(15)

and here V_{int} is the Kosterlitz-Thouless interaction, relegated to the role of the potential. This total Hamiltonian with the inclusion of the previously discussed total number N of investors as a further observable will comprise the constraints in the maximization of the entropy that we

will perform next. But first, let us simplify the interaction term in the potential again by averaging over the jth spins such that ($\pi \leq \theta \leq 0$ here)

$$
\begin{aligned}
V_{int_{mf}} &= -J \sum_{i,j}^{N} \frac{1}{\pi} \int_0^{\pi} cos(\theta_i - \theta_j) d\theta_j \\
&= -\frac{JN2}{\pi} \sum_i^{N} sin\,(\theta_i)\,.
\end{aligned}
\tag{16}
$$

again we can maximize the non-extensive entropy (per-investor) given the constraints of the potential and the first two central moments of the fluctuating variables of individual magnitude and bias such that for the ith investor we have

$$
\begin{aligned}
\langle S \rangle_q &= S_q = -\frac{1}{1-q}\left(1 - \int_0^{\pi} P(\theta,t)^q d\theta\right), \\
\left\langle v_{int_{mf}} \right\rangle_q &= \int_0^{\pi} [-\frac{2J}{\pi} sin\,(\theta) + Lmcos(\theta)] P(\theta,t)^q d\theta, \\
\left\langle (\theta - \overline{\theta}) \right\rangle_q &= \int_0^{\pi} [(\theta - \overline{\theta})] P(\theta,t)^q d\theta = 0, \\
\left\langle (\theta - \overline{\theta})^2 \right\rangle_q &= \int_0^{\pi} [(\theta - \overline{\theta})^2] P(\theta,t)^q d\theta
\end{aligned}
\tag{17}
$$

The maximum entropy method then states that we must maximize the entropy given the observables as constraints. This yields the following variation of the q-averaged observables

$$
\delta \langle S \rangle_q - \delta \left\{ \beta[\left\langle v_{int_{mf}} \right\rangle_q + \left\langle (\theta - \overline{\theta}) \right\rangle_q + \left\langle (\theta - \overline{\theta})^2 \right\rangle_q - \mu \langle N \rangle_q] \right\} \equiv 0.
\tag{18}
$$

The least biased PDF will again be of the non-extensive form and can be written as

$$
P(\theta,t) = \frac{\{1 + \beta(t)(q-1)[(\theta - \overline{\theta})^2 + [-\frac{2J}{\pi} sin\,(\theta) + Lmcos(\theta)] - \mu]\}^{\frac{-1}{q-1}}}{Z_q},
\tag{19}
$$

where the partition function is again related to the normalization and is now $Z_q = \int_0 \pi d\theta$

We wish to obtain the average bias and relate it to the price change. We then write the regular statistical average as M and utilize the market

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depth to obtain the correspondence

$$
\begin{aligned}
M &= \langle cos(\theta) \rangle = \int\limits_{0}^{\pi} cos(\theta)P(\theta,t)d\theta, \\
\langle \Delta x(t) \rangle &= \frac{\langle cos(\theta) \rangle}{\widetilde{\lambda}}.
\end{aligned}
$$

(20)

3   continuous model

We now wish to relax all of the approximations and the simplifications we built into the previous spin based models. To do this let us assume that the state vector for the system is comprised of two continuous degrees of freedom, the magnitude and the bias. Without restating the problem, let us write down the Hamiltonian for the investors in the two dimensions (y=magnitude,θ=bias 'angle') of magnitude and bias as H_o (denotes the non-interacting investor Hamiltonian). Also, let us propose some general interaction potential V the form of which will be examined subsequently . The important point here is that we are seeking to cast the problem of building a model for many interacting-investors into the powerful language of the many-particle physics as we feel this allows us to map some questions of modeling financial markets and investor behavior directly to well known physics-based paradigms. We have already touched upon this in our specific models discussed above, and now we wish to generalized the application of the technique.

The model will consist of the free and interacting parts of the Hamiltonian. That is, the ith investor will have $i(y(t),\sigma(t))$ magnitude of demand , with a bias of buy, sell or hold, at time t. The magnitude of demand can be assumed to be the demand for number of shares of a stock, and with the total number being considered fixed as a long-term constant $n = \int_0^{max} y(t)dy$. The direction of demand, is
discrete $(-1,0,+1)$ corresponding to buy, sell or hold. The Hamiltonian is

$$
H_i = \frac{D}{2}\overset{\bullet}{y}^2_{\sigma_i} + J\sum_j \sigma_j\sigma_i - K\sigma_i
$$

(21)

with the constraints of the moments

$$
\begin{aligned}
\langle y_{\sigma_i} - \bar{y}_{\sigma_i} \rangle &= \sum_0^N \int\limits_0^{max} \left(y_{\sigma_i} - \bar{y}_{\sigma_i} \right) P(y_{\sigma_i},\sigma_i,t)^q dy_{\sigma_i} \\
\langle (y_{\sigma_i} - \bar{y}_{\sigma_i})^2 \rangle &= \sum_0^N \int\limits_0^{max} \left(y_{\sigma_i} - \bar{y}_{\sigma_i} \right)^2 P(y_{\sigma_i},\sigma_i,t)^q dy_{\sigma_i}
\end{aligned}
$$

(22)

the maximization of the entropy then obtains the least biased distribution and with the normalization, $\sum \int P(\sigma_i,t)dy_{\sigma_i} = 1$ to unity. The extremization yields the least biased probability distribution function, $P(\sigma_i,t)$

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which upon taking the mean field approximation $\sum < \sigma_j >= M$ gives

$$P(y_{\sigma_i}, \sigma_i, t) = N_{\sigma_i}(t) \frac{1}{1 + \beta(t)(q-1)[(y_{\sigma_i} - \bar{y}_{\sigma_i})^2 - \alpha(t)JM\sigma_i + \gamma(t)K\sigma_i]^{\frac{1}{q-1}}}$$

.

$$(23)$$

The expectation value for the demand is the expression $< \sigma >= \sum_{\sigma,i} \int \sigma_i P(y_{\sigma_i}, \sigma_i, t) dy_{\sigma_i}$, which explicitly is

$$\langle \sigma \rangle = \sum_i^{N} \int_o^{max} \sigma_i P(y_{\sigma_i}, \sigma_i, t) dy_{\sigma_i}$$
$$= \int_0^{max} P(y_{+1}, \sigma_i = +1, t) dy_{+1} - \int_0^{max} P(y_{-1}, \sigma_i = -1, t) dy_{-1}$$

$$(24)$$

in terms of the number of shares demanded to be bought or sold or held for the moment at time t, the expectation value of excess demand is

$$= \frac{N_{+1}(t) - N_{-1}(t)}{N_{+1}(t) + N_{-1}(t) + N_0(t)},$$

$$(25)$$

This excess demand is proportional to the price and with the market depth λ as the proportionality factor we obtain the price and change in price if the definition of the variables of excess demand is relative $y = x - x'$

$$x(t) = \frac{\langle \sigma(t) \rangle}{\lambda},$$

$$(26)$$

This model is perhaps the most detailed of the three discussed . It is also the most robust and points the way to the inclusion of interaction terms that describe interactions observed factors in real world markets. These could include the independent investor, institutional investor clustering, floor trader and outside trader time lags, non-constant market depths, more complex herding behavior etc. These can all be included as observables multiplied by Lagrange multipliers as constraints in the maximization, and the derived least biased distributions though perhaps complicated can be solved numerically.

4   conclusion

The models presented here are of increasing complexity. However the three models, discrete and continuous can be generalized further and can easily be applied. The numbers of shares being bought, sold or held at any moment determine the instantaneous excess demand. This in turn is related to the instantaneous price or price change if relative variables

are used, by the market depth proportionality factor, here assumed constant. The statistical distributions of price changes and price have been shown to be well fitted by the students-T distribution and more recently the Tsallis nonex- tensive statistics distribution . The information theoretic approaches taken here assume the nonextensive entropy as a starting point, and obtains not surprisingly a power-law distribution for the numbers of shares to bought, sold or held. These are then summed and the time-dependent price change distributions obtained. A question to be answered in subsequent work is , what form of nonlinear interaction causes to arise a power-law distribution of price changes from a Gibbs-Boltzmann form of extensive entropy or information measure. How do nonlinear interactions modify the Gaussian distributions obtained from the extensive entropy. Also, how does the numerical simulation of the model, by Monte Carlo or stochastic trajectory methods, compare to actual market data price changes. Previous numerical simulations of power-law distributions have been shown by us to fit very well the high frequency stochastic time series data of price changes of the S&P500 and we have also applied the power-law statistics to generalized Black-Scholes equations of prices for options and derivatives secondary markets for the underlying stocks and stocks indices etc. We expect the present models especially the power-law distributed model to fit accurately the market price data , and given the parameters extracted from market data, the model can be a theory that begins to describe the statistical uncertainties, and on the average the microscopic interactions that occur between investors in a financial market and which then the theoretical model can subsequently be utilized for minimization of risk and the design of investment strategies in economics and finance. The author wishes to state that most of this manuscript was written in 20 Nov 2001 with M.D. Johnson and John Evans at the University of central Florida Physics Department, Orlando, Florida. Recent research of the author's has refocused on this area of research, and the publication follows.

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